Slowing time: Markov-modulated Brownian motion with a sticky boundary

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Abstract

We analyze the stationary distribution of regulated Markov modulated Brownian motions (MMBM) modified so that their evolution is slowed down when the process reaches level zero — level zero is said to be *sticky*. To determine the stationary distribution, we extend to MMBMs a construction of Brownian motion with sticky boundary, and we follow a Markov-regenerative approach similar to the one developed in past years in the context of quasi-birth-and-death processes and fluid queues. We also rely on recent work showing that Markov-modulated Brownian motions may be analyzed as limits of a parametrized family of fluid queues. We use our results to revisit the stationary distribution of the well-known regulated MMBM.

Keywords: Fluid queues, regenerative processes, Markov-modulated Brownian motion, sticky boundary.

1 Introduction

Systems in real life are designed with feedback loops: a buffer in a telecommunication network is not allowed to repeatedly overflow without its input being throttled, water conservation measures are taken before reservoirs get thoroughly dry, and so on. This is our reason for being interested in stochastic processes with *reactive* boundaries, that is, processes that change behaviour

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upon hitting some boundary. In the present paper, we focus on regulated Markov modulated Brownian motions (regulated MMBMs for short), with a *sticky boundary* at level 0.

An MMBM is a two-dimensional process $\{X(t), \varphi(t) : t \geq 0\}$ with $X(\cdot) \in \mathbb{R}$ and $\varphi(\cdot) \in \mathcal{M} = \{1, \dots, m\}$ with $m < \infty$. The component $\{\varphi(t)\}$ is a continuous-time Markov chain and controls as follows the evolution of $\{X(t)\}$:

$$X(t) = \int_0^t \mu_{\varphi(s)} \, \mathrm{d}s + \int_0^t \sigma_{\varphi(s)} \, \mathrm{d}W(s).$$

Here, μ_i and σ_i, for $i \in \mathcal{M}$, are real numbers with $\sigma_i \geq 0$, and $\{W(t)\}$ is a standard Brownian motion independent of $\{\varphi(t)\}$. We call $\varphi(t)$ the phase at time t and $X(t)$ the level.

The process $\{Z(t), \varphi(t)\}$ regulated at zero is

$$Z(t) = X(t) + |\inf_{0 \leq s \leq t} X(s)|. \tag{1}$$

We assume that the MMBM is drifting to $-\infty$, so that $\{Z(t), \varphi(t)\}$ has a stationary distribution; this is made more precise in Section 3 and we refer to Asmussen [1], Rogers [18] for a general presentation of basic properties. If $\sigma_i = 0$ for all i, $\{Z(t), \varphi(t)\}$ is a fluid queue, a family of processes extensively analyzed by Ramaswami [16], da Silva Soares and Latouche [7] and Bean *et al.* [3], among others.

Brownian motions with a sticky boundary were introduced by Feller [8] in the 1950s. Briefly stated, the regulated Brownian motion is slowed down when it is at level 0, in such a way that, without actually staying at zero for any interval of time of positive length, it does spend in that level an amount of time with positive Lebesgue measure.

The construction in Harrison and Lemoine [10] works as follows: one starts with a Brownian motion $\{X^*(t) : t \geq 0\}$ with parameters μ and σ^2, define its regulator $R^*(t) = |\inf_{0 \leq s \leq t} X^*(s)|$ and define the regulated process as $Z^*(t) = X^*(t) + R^*(t)$. Next, one defines the functions $V^*(t) = t + R^*(t)/\omega$, where $\omega > 0$ is some fixed constant, and $\Gamma^*(t)$ such that $V^*(\Gamma^*(t)) = t$. Finally, one defines

$$Y^*(t) = Z^*(\Gamma^*(t)). \tag{2}$$

We refer to Γ^* as the new clock. The process $\{Y^*(t)\}$ is a Brownian motion with sticky boundary, with parameters μ, σ^2, and ω. The time change in (2) modifies each trajectory of Z^* by increasing the time spent in state 0, while leaving $\{Y^*(t)\}$ to behave exactly like $\{Z^*(t)\}$ away from zero.

We extend in two ways this construction to Markov modulated Brownian motion. First, we define in Section 4 a straightforward generalization based

on the regulator $R(t)$. Taking the phase into account, we decompose $R(t)$ as the sum of m sub-regulators

$$r_i(t) = \int_0^t 1\{\varphi(s) = i\}\,\mathrm{d}R(s) \tag{3}$$

for $i \in \mathcal{M}$ and all t. It is clear that $r_i(t)$ increases only when $Z(t) = 0$ and $\varphi(t) = i$, and that $R(t) = \sum_{i\in\mathcal{M}} r_i(t)$. Next, we define

$$V(t) = t + \sum_{i\in\mathcal{M}} r_i(t)/\omega_i \tag{4}$$

where $\omega_i > 0$, and we define the function $\Gamma(t)$ such that $V(\Gamma(t)) = t$. Observe that through the definition (4) of $V(t)$, we allow the clock $\Gamma(t)$ to slow down at different rates for different phases. Our new process is $\{Y(t), \bar\varphi(t)\}$ with $Y(t) = Z(\Gamma(t))$, $\bar\varphi(t) = \varphi(\Gamma(t))$. In Section 7 we investigate another process, such that the marginal distribution of the phase is allowed to change as a result of the process hitting the boundary.

To determine the stationary distribution of our processes, we follow a Markov-regenerative approach. We choose points of regeneration forming a subset of the epochs when the process hits level 0: let $\{\Delta_n, n \geq 0\}$ denote a sequence of i.i.d. random variables exponentially distributed with parameter q. We define

$$\theta_{n+1} = \inf\{t > \theta_n + \Delta_{n+1} : Y(t) = 0\}, \tag{5}$$

for $n \geq 0$, with $\theta_0 = 0$. In short, once the process hits the boundary, we start an exponential timer and we do not register the instantaneous returns to 0 by the Brownian motion; at the expiration of the timer, one is again able to register the next hit at zero. The process $\{\bar\varphi_n\}$ embedded at the regeneration epochs, with $\bar\varphi_n = \bar\varphi(\theta_n)$, for $n \geq 1$, is an irreducible discrete-time Markov chain with stationary distribution $\boldsymbol{\rho}$.

We also define

$$M_{ij}(x) = \mathrm{E}[\int_{\theta_n}^{\theta_{n+1}} 1\{Y(s) \in [0,x], \bar\varphi(s) = j\}\,\mathrm{d}s | \bar\varphi_n = i]$$

independently of n; that is, $M_{ij}(x)$ is the expected sojourn time of $\{Y(t), \bar\varphi(t)\}$ in $[0,x] \times \{j\}$ during an inter-regeneration interval, given that the phase is i at the beginning of the interval. The components of $\boldsymbol{m}$, defined as $\boldsymbol{m} = M(\infty)\mathbf{1}$, are the conditional expected lengths of intervals between regeneration points, given the phase at the last regeneration: its ith component is $m_i = \mathrm{E}[\theta_{n+1} - \theta_n | \bar\varphi_n = i]$. Finally, with

$$G_i(x) = \lim_{t\to\infty} \mathrm{P}[Y(t) \leq x, \bar\varphi(t) = i],$$

and $\boldsymbol{G}(x) = \begin{bmatrix} G_1(x) & \dots & G_m(x) \end{bmatrix}$, we have

$$\boldsymbol{G}(x) = (\boldsymbol{\rho m})^{-1}\, \boldsymbol{\rho} M(x) \tag{6}$$

(see Çinlar [5, Section 10.7]). With our choice (5) for the regeneration points, $\boldsymbol{\rho}$ and M are necessarily functions of the parameter q, while the function $\boldsymbol{G}$ is independent of q; as we show in Theorem 5.3, the expression in the right-hand side of (6) is indeed independent of q. We note that any set of regeneration points will do, provided that they lead to a discrete-time Markov chain and that one is able to determine the expected sojourn times between regenerations.

We rely on results obtained in Latouche and Nguyen [13, 14] to determine $\boldsymbol{\rho}$ and M. We defined in [14] a family of fast oscillating fluid queues and we showed that MMBMs arise as limits of such fluid queues as the speed of oscillation increases to infinity. We give in Section 3 the basic definition of the approximating fluid queues as well as some properties that we shall be using throughout the paper. Before that, we show in Section 2 how to construct a family of fast oscillating fluid queues that converge to a Brownian motion with sticky boundary.

We determine in Section 4 the stationary distribution of our first family of Markov modulated processes with sticky boundary. In Section 5, we apply our regenerative approach to the well-known regulated MMBM with one boundary and obtain a new form for its stationary distribution; this is discussed in Section 6, where we analyze the physical meaning of our result and compare it to other expressions available from the literature. In Section 7, we define and analyze our second model for MMBMs with sticky boundary, and we give some brief concluding remarks in Section 8.

Notation We represent by $\mathbf{1}$ a column vector of 1s and by $\mathbf{0}$ a vector of 0s. We generally use the notation X for unregulated processes, Z for processes with one boundary, and Y for processes with a sticky boundary. Also, a bar over a symbol, as in $\bar{\varphi}$, indicates the phase process of the sticky version.

2 Sticky Brownian motion

As indicated in the introduction, we proceed in a manner similar to Harrison and Lemoine [10]. In this section, $\{X(t)\}$ denotes a Brownian motion with parameters $\mu < 0$ and $\sigma^2 > 0$, and the regulated process is $Z(t) = X(t) + R(t)$, where $R(t) = |\inf_{0 \le s \le t} X(s)|$. Our objective is to create a process that behaves exactly like Z when it is strictly positive but spends more time at 0.

To do this, one defines the function $V(t) = t + R(t)/\omega$, where $\omega > 0$, and the function $\Gamma(t)$ such that $V(\Gamma(t)) = t$. Finally, one defines

$$Y(t) = Z(\Gamma(t)).$$

The process $\{Y(t)\}$ is a sticky Brownian motion with parameters μ, σ^2 and ω.

The process $R(t)$ is non-negative, non-decreasing and continuous, and remains constant when $Z(t) > 0$. Therefore, $V(t)$ is strictly increasing and continuous, and $\Gamma(t)$ is well-defined, continuous and strictly increasing. The purpose of $\Gamma(t)$ is to serve as a new clock, which increases at the same rate as t when $Z(t) > 0$, and at a slower rate when $Z(t) = 0$, that is, when $R(t)$ is increasing. The stationary distribution of Y is

$$\lim_{t\to\infty} \mathrm{P}[Y(t) \le x] = \frac{|\mu|}{|\mu| + \omega} + \frac{\omega}{|\mu| + \omega}(1 - e^{2\mu x/\sigma^2}) \tag{7}$$

([10, p. 221]), notice that it has a mass at zero.

As an approximation to $\{X(t)\}$, we define a family of two-state fluid queues $\{X_\lambda(t), \kappa_\lambda(t)\}$ indexed by λ. The generator of the phase process $\{\kappa_\lambda(t)\}$ is

$$T = \begin{bmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{bmatrix}$$

and the fluid rates are $c_1 = \mu + \sigma\sqrt{\lambda}$ and $c_2 = \mu - \sigma\sqrt{\lambda}$. It is shown in Ramaswami [16] that $\{X_\lambda(t)\}$ converges weakly to $\{X(t)\}$ as λ tends to ∞. We use the term *flip-flop* processes to characterize $\{X_\lambda(t)\}$ and other fluid queues to be defined in later sections, as a shorthand reminder of their behavior: the fluid queue switches steadily faster, as λ increases, between two increasing fluid rates. We use the regulator $R_\lambda(t) = |\inf_{0\le s\le t} X_\lambda(s)|$ and the regulated process $Z_\lambda(t) = X_\lambda(t) + R_\lambda(t)$ to define a two-state flip-flop fluid queue $\{Y_\lambda(t)\}$ with a sticky boundary at zero.

The total time spent by $Z_\lambda(t)$ at level 0 is

$$L^0_\lambda(t) = \int_0^t 1\{Z_\lambda(s) = 0\}\,\mathrm{d}s.$$

It is a sum of individual intervals, each of which is exponentially distributed with parameter λ. To define the process $\{Y_\lambda(t)\}$, we change the behaviour of the phase when the level is 0, and we assume that the intervals of time spent there are now exponentially distributed with parameter $a\sqrt{\lambda}$ instead of λ, for some $a > 0$. Equivalently, the intervals of time spent at level 0 are stretched by a factor $\sqrt{\lambda}/a$. The function that defines the new time is

$$V_\lambda(t) = t - L^0_\lambda(t) + L^0_\lambda(t)\frac{\sqrt{\lambda}}{a}.$$

Clearly, $L^0_\lambda(t) = R_\lambda(t)/(\sigma\sqrt{\lambda} - \mu)$, so that

$$V_\lambda(t) = t + R_\lambda(t)\frac{\sqrt{\lambda} - a}{a(\sigma\sqrt{\lambda} - \mu)}. \tag{8}$$

Finally, the new clock is given by the function $\Gamma_\lambda(t)$ such that $V_\lambda(\Gamma_\lambda(t)) = t$, and the fluid queue with sticky boundary is $\{Y_\lambda(t), \bar{\kappa}_\lambda(t)\}$, with $Y_\lambda(t) = Z_\lambda(\Gamma_\lambda(t))$ and $\bar{\kappa}_\lambda(t) = \kappa_\lambda(\Gamma_\lambda(t))$.

Theorem 2.1 *The processes $\{Y_\lambda(t)\}$ weakly converge to the sticky Brownian motion $\{Y(t)\}$ with parameters μ, σ^2 and $\omega = a\sigma$.*

Proof To simplify our presentation, we rewrite (8) as $V_\lambda(t) = t + R_\lambda(t)a_\lambda$. We also rewrite the equation $V_\lambda(\Gamma_\lambda(t)) = t$ as

$$\Gamma_\lambda(t) + R_\lambda(\Gamma_\lambda(t))a_\lambda = t. \tag{9}$$

We know by [14, Corollary 3.3] that $\{Z_\lambda(t)\}$ weakly converges to $\{Z(t)\}$ as $\lambda \to \infty$, and so $\{R_\lambda(t)\}$ weakly converges to $\{R(t)\}$. In addition, the coefficient of $R_\lambda(t)$ in (8) converges to $1/(a\sigma)$. Therefore, the finite-dimensional distribution of $\{Y_\lambda(t)\}$ converges to the finite-dimensional distribution of the Brownian motion $\{Y(t)\}$ with a sticky boundary and parameters μ, σ^2 and $\omega = a\sigma$, and we need only to prove tightness.

By Billingsley [4, Theorem 7.3] we need to prove that the processes are tight at time 0, which is obvious since they are all equal to zero, and that for all ε, η, there exist $\delta^* > 0$ and λ^* such that

$$\mathrm{P}[\sup_{|s-t|\le\delta} |Y_\lambda(t) - Y_\lambda(s)| \ge \varepsilon] \le \eta$$

for all $\lambda > \lambda^*$ and $\delta < \delta^*$. Now, assume temporarily that $s \le t$. We have by (9)

$$\begin{aligned}\Gamma_\lambda(t) - \Gamma_\lambda(s) &= t - s - (R_\lambda(\Gamma_\lambda(t)) - R_\lambda(\Gamma_\lambda(s)))a_\lambda \\ &\le t - s,\end{aligned}$$

since both R_λ and Γ_λ are non-decreasing functions and $a_\lambda > 0$ for sufficiently large λ. The reverse inequality holds if $s \ge t$ and so $|\Gamma_\lambda(t) - \Gamma_\lambda(s)| \le |t - s|$ for all s, t. Therefore,

$$\begin{aligned}\sup_{|s-t|\le\delta} |Y_\lambda(t) - Y_\lambda(s)| &= \sup_{|s-t|\le\delta} |Z_\lambda(\Gamma_\lambda(t)) - Z_\lambda(\Gamma_\lambda(s))| \\ &\le \sup_{|s-t|\le\delta} |Z_\lambda(t) - Z_\lambda(s)|\end{aligned}$$

and

$$\mathrm{P}[\sup_{|s-t|\le\delta} |Y_\lambda(t) - Y_\lambda(s)| \ge \varepsilon] \le \mathrm{P}[\sup_{|s-t|\le\delta} |Z_\lambda(t) - Z_\lambda(s)| \ge \varepsilon].$$

This completes the proof since we know by Ramaswami [17, Theorem 5] and Whitt [19, Corollary 7] that $\{Z_\lambda(t)\}$ is tight. $\square$

Remark 2.2 It is necessary that the transition rate of κ_λ at level zero should grow like $\sqrt{\lambda}$. To see this, let us assume that we use some general function $f(\lambda)$ instead of $a\sqrt{\lambda}$. The stretching factor is $\lambda/f(\lambda)$ and (8) becomes

$$\begin{aligned}
V_\lambda(t) &= t + L^0_\lambda(t)(\frac{\lambda}{f(\lambda)} - 1) \\
&= t + R_\lambda(t)\frac{\lambda - f(\lambda)}{(\sigma\sqrt{\lambda} - \mu)f(\lambda)} \\
&= t + R_\lambda(t)\frac{1 - f(\lambda)/\lambda}{(\sigma - \mu/\sqrt{\lambda})f(\lambda)/\sqrt{\lambda}}.
\end{aligned}$$

If $f(\lambda)/\sqrt{\lambda} \to 0$ as $\lambda \to \infty$, then $V_\lambda(t) \to \infty$ for all t such that $R_\lambda(t) > 0$, and the limit of $\{Y_\lambda(t)\}$ is 0 for all t. On the contrary, if $f(\lambda)/\sqrt{\lambda} \to \infty$, then $V_\lambda(t) \to t$, $Y_\lambda(t) \to Z(t)$, and the limiting process is just the original regulated Brownian motion.

3 Preliminaries

We assume in the sequel that $\{\varphi(t)\}$ is an irreducible Markov process with generator Q and we make the following two assumptions.

Assumption 3.1 *The variances σ_i are strictly positive for all i in $\mathcal{M}$.*

This assumption allows us to significantly simplify our presentation.

Assumption 3.2 *The stationary drift $\boldsymbol{\alpha\mu}$ is strictly negative, where $\boldsymbol{\mu} = \begin{bmatrix}\mu_1 & \dots & \mu_m\end{bmatrix}$ and $\boldsymbol{\alpha}$ is the stationary probability vector of $\{\varphi(t)\}$, that is, $\boldsymbol{\alpha}Q = \mathbf{0}$, $\boldsymbol{\alpha}\mathbf{1} = 1$.*

The inequality $\boldsymbol{\alpha\mu} < 0$ is the necessary and sufficient condition for our regulated MMBMs to have a stationary probability distribution.

We shall use the same approach as in [14] and start our analysis from a parametrized family of approximating fluid queues driven by a two-dimensional phase process $\{\kappa_\lambda(t), \varphi_\lambda(t)\}$ on the state space $\{1, 2\} \times \mathcal{M}$, with generator

$$Q^*_\lambda = \begin{bmatrix} Q - \lambda I & \lambda I \\ \lambda I & Q - \lambda I \end{bmatrix} \tag{10}$$

and fluid rate matrix

$$C^*(\lambda) = \begin{bmatrix} D + \sqrt{\lambda}\Theta & \\ & D - \sqrt{\lambda}\Theta \end{bmatrix},$$

where $D = \mathrm{diag}(\mu_1, \ldots, \mu_m)$ and $\Theta = \mathrm{diag}(\sigma_1, \ldots, \sigma_m)$. Next, we define

$$X_\lambda(t) = \int_0^t C^*_{\kappa_\lambda(s), \varphi_\lambda(s)}(\lambda)\, \mathrm{d}s$$

and $Z_\lambda(t) = X_\lambda(t) + |\inf_{0 \le s \le t} X_\lambda(s)|$. For λ large enough, the rates $\mu_i + \sigma_i\sqrt{\lambda}$ corresponding to $\kappa_\lambda = 1$ are all positive, and the rates $\mu_i - \sigma_i\sqrt{\lambda}$ corresponding to $\kappa_\lambda = 2$ are all negative. It is shown in [14] that the processes $\{Z_\lambda(t), \varphi_\lambda(t)\}$ weakly converge to the MMBM $\{Z(t), \varphi(t)\}$ as $\lambda \to \infty$. In this paper, the phase process $\{\kappa_\lambda(t), \varphi_\lambda(t)\}$ will define the evolution of our regulated processes whenever the fluid level is strictly positive. Different rules will apply at level 0 for different processes, and will be separately detailed in each case.

We use the same definition (5) for the regeneration points of all processes $\{Z_\lambda(t), \kappa_\lambda(t), \varphi_\lambda(t)\}$ and we omit to indicate that the θ_ns depend on λ, so as not to clutter the notation unduly. A key quantity for the analysis of fluid queues is the matrix of first return probabilities to level 0, starting from level 0 in a phase with strictly positive fluid rate. Because different processes have different behaviors at level 0 but the same behavior away from the boundary, it will be useful to use the sequence $\{\tau_n\}$ of first instants when the fluid starts increasing away from level 0 after a regeneration epoch:

$$\tau_n = \inf\{t > \theta_n : \kappa_\lambda(t) = 1\},$$

for $n \ge 0$. During an interval (τ_n, θ_{n+1}), we need to know at any given time whether the current timer has expired or not. For that reason, we add a third phase component, named χ_λ, with $\chi_\lambda = 1$ if the timer has not expired yet, and $\chi_\lambda = 2$ otherwise. The two-dimensional phase $(\kappa_\lambda, \varphi_\lambda)$ always evolves according to the transition matrix (10) and the transitions of $(\kappa_\lambda, \chi_\lambda, \varphi_\lambda)$ are controlled, during an interval (τ_n, θ_{n+1}), by the matrix

$$T(\lambda) = \left[\begin{array}{cc|cc} Q - \lambda I - qI & qI & \lambda I & \\ & Q - \lambda I & & \lambda I \\ \hline \lambda I & & Q - \lambda I - qI & qI \\ & \lambda I & & Q - \lambda I \end{array}\right]. \tag{11}$$

At regeneration times, the new component χ_λ instantaneously switches from $\chi_\lambda(\theta_n^-) = 2$ to $\chi_\lambda(\theta_n^+) = 1$, where $\chi_\lambda(\theta_n^-) = \lim_{t \uparrow \theta_n} \chi_\lambda(t)$ and $\chi_\lambda(\theta_n^+) =$

$\lim_{t\downarrow\theta_n}\chi_\lambda(t)$. The diagonal matrix of fluid rates is

$$C(\lambda)=\begin{bmatrix} I_2\otimes(D+\Theta\sqrt{\lambda}) & \\ & I_2\otimes(D-\Theta\sqrt{\lambda})\end{bmatrix}$$

where I_2 is the identity matrix of order 2, and we partition the state space into the subsets

$$\mathcal{S}_+=\{(1,\chi_\lambda,\varphi_\lambda):\chi_\lambda\in\{1,2\},\varphi_\lambda\in\mathcal{M}\}$$

and

$$\mathcal{S}_-=\{(2,\chi_\lambda,\varphi_\lambda):\chi_\lambda\in\{1,2\},\varphi_\lambda\in\mathcal{M}\}.$$

We similarly partition the matrix $T(\lambda)$ as

$$T(\lambda)=\begin{bmatrix} T_{++}(\lambda) & T_{+-}(\lambda)\\ T_{-+}(\lambda) & T_{--}(\lambda)\end{bmatrix}$$

and we write $C_+=I_2\otimes(D+\Theta\sqrt{\lambda})$ and $C_-=I_2\otimes(D-\Theta\sqrt{\lambda})$.

The matrix of first return probabilities, indexed by $\mathcal{S}_+\times\mathcal{S}_-$, is denoted as Υ_λ and defined by

$$\begin{aligned}(\Upsilon_\lambda)_{k,i;k',j}=\mathrm{P}[\xi<\infty,\chi_\lambda(\xi)=k',\varphi_\lambda(\xi)=j\\ |Z_\lambda(0)=0,\kappa_\lambda(0)=1,\chi_\lambda(0)=k,\varphi_\lambda(0)=i]\end{aligned}\tag{12}$$

where $\xi=\inf\{t>0:Z_\lambda(t)=0\}$ is the first return time to level 0. It is well known (Rogers [18]) that Υ_λ is the minimal non-negative solution of the Riccati equation

$$C_+^{-1}T_{+-}(\lambda)+C_+^{-1}T_{++}(\lambda)X+X|C_-|^{-1}T_{--}(\lambda)+X|C_-|^{-1}T_{-+}(\lambda)X=0.$$

One easily verifies that it has the structure

$$\Upsilon_\lambda=\begin{bmatrix}\Psi_\lambda(q) & \Psi_\lambda^c(q)\\ 0 & \Psi_\lambda\end{bmatrix}$$

where

- $\Psi_\lambda(q)$ is the probability matrix of returning to the original level *before* the exponential timer expires, it is the minimal non-negative solution of

$$\begin{aligned}\lambda(D+&\sqrt{\lambda}\Theta)^{-1}+(D+\sqrt{\lambda}\Theta)^{-1}(Q-\lambda I-qI)X\\ &+X|D-\sqrt{\lambda}\Theta|^{-1}(Q-\lambda I-qI)+\lambda X|D-\sqrt{\lambda}\Theta|^{-1}X=0,\end{aligned}$$

- $\Psi_\lambda = \Psi_\lambda(0)$ is the return probability matrix without any time constraint, and
- $\Psi^c_\lambda(q)$ is the probability of returning to the original level *after* the exponential timer has expired, so that

$$\Psi^c_\lambda(q) = \Psi_\lambda - \Psi_\lambda(q). \tag{13}$$

At level 0, we need two transition matrices. The first matrix has entries

$$(P_{\lambda,0})_{ij} = \mathrm{P}[\tau_n > \theta_n + \Delta_{n+1}, \varphi_\lambda(\theta_n + \Delta_{n+1}) = j | \varphi_\lambda(\theta_n) = i], \tag{14}$$

for i, j in $\mathcal{M}$: these are the probabilities that the process continuously remains at level 0 until the expiration of the exponential timer, at which time $\varphi_\lambda = j$, given the phase at time θ_n is i. We need not specify the other phase component as $\kappa_\lambda(\theta_n + \Delta_{n+1}) = 2$. The second matrix is

$$(P_{\lambda,1})_{ij} = \mathrm{P}[\tau_n < \theta_n + \Delta_{n+1}, \varphi_\lambda(\tau_n) = j | \varphi_\lambda(\theta_n) = i], \tag{15}$$

for i, j in $\mathcal{M}$: the exponential timer has not yet gone off at time τ_n, the component κ_λ switches from 2 to 1 and the component χ_λ remains equal to 1.

The phase transition matrix at regenerative epochs is Φ_λ, with

$$(\Phi_\lambda)_{ij} = \mathrm{P}[\varphi_\lambda(\theta_{n+1}) = j | \varphi_\lambda(\theta_n) = i].$$

Again, we do not need to specify the remaining components of the phases: $\kappa_\lambda(\theta_n) = 2$ since the fluid rate is negative at that time and $\chi_\lambda(\theta_n^+) = 1$ since a new timer interval begins immediately after the regeneration.

Lemma 3.3 *The transition matrix at the regeneration epochs is given by*

$$\Phi_\lambda = I - (I - P_{\lambda,1}\Psi_\lambda(q))^{-1}(I - P_{\lambda,0} - P_{\lambda,1}\Psi_\lambda), \tag{16}$$

Its stationary probability vector $\boldsymbol{\rho}_\lambda$ is

$$\boldsymbol{\rho}_\lambda = c\boldsymbol{\nu}_\lambda(I - P_{\lambda,1}\Psi_\lambda(q)) \tag{17}$$

for some scalar c, where $\boldsymbol{\nu}_\lambda$ is such that

$$\boldsymbol{\nu}_\lambda(P_{\lambda,0} + P_{\lambda,1}\Psi_\lambda) = \boldsymbol{\nu}_\lambda, \qquad \boldsymbol{\nu}_\lambda\mathbf{1} = 1. \tag{18}$$

Proof The transition matrix satisfies the following equation,

$$\Phi_\lambda = P_{\lambda,0} + P_{\lambda,1}\Psi^c_\lambda(q) + P_{\lambda,1}\Psi_\lambda(q)\Phi_\lambda. \tag{19}$$

Indeed, at time θ_n the process is at level 0 and either it does not leave level 0 before the timer expires (this corresponds to the first term), or it does leave level zero and returns after the timer has expired (this is the second term) or it leaves level 0 and returns before the timer has expired, in which case we still have to wait for the next regeneration point (this gives the third term).

Now, starting from level 0 and any phase in $\mathcal{S}_+$, there is a strictly positive probability that the process returns to level 0 after the timer has expired. Thus, $\Psi^c_\lambda(q)\mathbf{1} > \mathbf{0}$ or equivalently, $\Psi_\lambda(q)\mathbf{1} < \mathbf{1}$, so that $P_{\lambda,1}\Psi_\lambda(q)$ is a strictly sub-stochastic matrix and $I - P_{\lambda,1}\Psi_\lambda(q)$ is non-singular. Thus, (19) becomes

$$\Phi_\lambda = (I - P_{\lambda,1}\Psi_\lambda(q))^{-1}(P_{\lambda,0} + P_{\lambda,1}\Psi^c_\lambda(q))$$

by (13). We may rewrite the last equation as

$$\Phi_\lambda = (I - P_{\lambda,1}\Psi_\lambda(q))^{-1}(P_{\lambda,0} + P_{\lambda,1}(\Psi_\lambda - \Psi_\lambda(q)))$$

and (16) is proved.

The stationary probability vector $\boldsymbol{\rho}_\lambda$ is such that $\boldsymbol{\rho}_\lambda\Phi_\lambda = \boldsymbol{\rho}_\lambda$, or

$$\boldsymbol{\rho}_\lambda(I - P_{\lambda,1}\Psi_\lambda(q))^{-1}(I - P_{\lambda,0} - P_{\lambda,1}\Psi_\lambda) = \mathbf{0}, \tag{20}$$

which proves (17) as soon as we show that $\boldsymbol{\nu}_\lambda$ exists and is unique.

By Assumption 3.2, the return time to level 0 is finite a.s., so that $\Psi_\lambda\mathbf{1} = \mathbf{1}$ for all λ and $P_{\lambda,0} + P_{\lambda,1}\Psi_\lambda$ is an irreducible stochastic matrix, with a unique stationary probability vector $\boldsymbol{\nu}_\lambda$. This concludes the proof. □

Remark 3.4 It is obvious that the stationary distribution of the phase at epochs of regeneration depends, through the transition matrices $P_{\lambda,0}$ and $P_{\lambda,1}$, on the rules of evolution of the phase when the fluid is at level 0. In order to avoid confusion, we use in the sequel the notation $\boldsymbol{\rho}_\lambda$ and its limit $\boldsymbol{\rho}$ for the process associated with the MMBM with sticky boundary, $\boldsymbol{\rho}^*_\lambda$ (with limit $\boldsymbol{\rho}^*$) for the flip-flop process analyzed in Section 5 and associated with the traditional MMBM, and $\widetilde{\boldsymbol{\rho}}$ for the process analysed in Section 7, with sticky boundary and resampling of the phase at level 0.

4 MMBM with sticky boundary

As explained in the introduction, we start from the function $V(t)$ defined in (4), which is monotone and continuous. We use it to define the new clock

Γ such that $V(\Gamma(t)) = t$, and to define the new process $\{Y(t), \bar{\varphi}(t)\}$ such that $Y(t) = Z(\Gamma(t))$ and $\bar{\varphi}(t) = \varphi(\Gamma(t))$.

We call this process a Markov-modulated Brownian motion with sticky boundary, with parameters $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{\omega}$ and Q. To obtain its stationary distribution, we proceed in three steps: we construct a family of approximating fluid queues, then we determine the stationary distribution $\boldsymbol{\rho}$ at epochs of regeneration, and finally we obtain the matrix $M(x)$ of expected time spent in $[0, x]$ during regeneration intervals.

We decompose the regulator $R_\lambda(t) = |\inf_{0\le s\le t} X_\lambda(s)|$ of the flip-flop fluid queue $\{X_\lambda(t), \kappa_\lambda(t), \varphi_\lambda(t)\}$ into its sub-regulators

$$r_{\lambda,i}(t) = \int_0^t 1\{\varphi_\lambda(s) = i\}\, \mathrm{d}R_\lambda(s), \tag{21}$$

and we write Z_λ as the sum $Z_\lambda = X_\lambda + \sum_i r_{\lambda,i}$. The total time spent by the process in phase i during the interval $(0, t)$ is $U_{\lambda,i}(t) = \int_0^t 1\{\varphi(s) = i\}\, \mathrm{d}s$, for $i \in \mathcal{M}$.

We repeat for every phase the argument in Section 2: the time spent by $\{Z_\lambda(t)\}$ at level 0 in phase i is $r_{\lambda,i}(t)/|\mu_i - \sigma_i\sqrt{\lambda}|$ and the functions that redefine time are

$$\begin{aligned} V_{\lambda,i}(t) &= U_{\lambda,i}(t) - r_{\lambda,i}(t)\frac{1}{|\mu_i - \sigma_i\sqrt{\lambda}|} + r_{\lambda,i}(t)\frac{1}{|\mu_i - \sigma_i\sqrt{\lambda}|}\frac{\lambda + |Q_{ii}|}{a_i\sqrt{\lambda} + |Q_{ii}|} \\ &= U_{\lambda,i}(t) + r_{\lambda,i}(t)\frac{\lambda - a_i\sqrt{\lambda}}{(\sigma_i\sqrt{\lambda} - \mu_i)(a_i\sqrt{\lambda} + |Q_{ii}|)}, \end{aligned}$$

for i in $\mathcal{M}$. For λ large enough, their sum is

$$V_\lambda(t) = t + \sum_i r_{\lambda,i}(t)\frac{\lambda + O(\sqrt{\lambda})}{a_i\sigma_i\lambda + O(\sqrt{\lambda})}$$

and converges to $V(t)$ defined in (4) with $\omega_i = a_i\sigma_i$. In matrix notation, we may write

$$V(t) = t + \boldsymbol{r}(t)A^{-1}\Theta^{-1}\mathbf{1}, \tag{22}$$

where $A = \mathrm{diag}(a_1, \ldots a_m)$ and $a_i > 0$ for all i. With the function $V_\lambda(t)$, we define the new clock Γ_λ such that $V_\lambda(\Gamma_\lambda(t)) = t$, and the new processes $\{Z_\lambda(\Gamma_\lambda(t)), \kappa_\lambda(\Gamma_\lambda(t)), \varphi_\lambda(\Gamma_\lambda(t)))$. By [14, Theorem 2.7], $\{Z_\lambda(t), \varphi_\lambda(t)\}$ weakly converge to $\{Z(t), \varphi(t)\}$ and, by the continuity property of $\Gamma(t)$, it is clear that the finite-dimensional distribution of $\{Z_\lambda(\Gamma_\lambda(t)), \varphi_\lambda(\Gamma_\lambda(t))\}$ converge to the finite-dimensional distribution of $\{Y(t), \bar{\varphi}(t)\}$.

The process $\{Z_\lambda(\Gamma_\lambda(t)), \kappa_\lambda(\Gamma_\lambda(t)), \varphi_\lambda(\Gamma_\lambda(t))\}$ is not very convenient, however, as its definition does not conform to the usual parametrization of

fluid queues. For that reason, we define another fluid queue, denoted as $\{Y_\lambda(t), \bar{\kappa}_\lambda(t), \bar{\varphi}_\lambda(t)\}$. This new process and $\{Z_\lambda(\Gamma_\lambda(t)), \kappa_\lambda(\Gamma_\lambda(t)), \varphi_\lambda(\Gamma_\lambda(t))\}$ are not pathwise identical but they have the same distribution.

The two-dimensional phase $(\bar{\kappa}_\lambda, \bar{\varphi}_\lambda)$ is controlled by the generator Q^*_λ defined in (10) as long as Y_λ is strictly positive. When $Y_\lambda = 0$, the transition rates are given by the new matrix

$$Q_{\lambda,0} = \left[\sqrt{\lambda}A \quad (1/\sqrt{\lambda})A(Q - \lambda I)\right]. \tag{23}$$

This means that while the process is in level 0,

- if $\varphi_\lambda = i$ and $\kappa_\lambda = 2$, intervals of time are stretched by a factor $\sqrt{\lambda}/a_i$,
- there may be a change to $\varphi_\lambda = j$ with the new rate $\bar{Q}_{ij} = a_i Q_{ij}/\sqrt{\lambda}$ without changing κ_λ,
- or the process may change to $\kappa_\lambda = 1$ at the rate $a_i\sqrt{\lambda}$, and leave the level 0, without changing φ_λ.

We denote by $\boldsymbol{\rho}_\lambda$ the stationary distribution of $\bar{\varphi}_\lambda$ at epochs of regeneration and by $M_\lambda(x)$ the matrix of conditional expected time spent by the fluid queue in $[0, x]$.

Lemma 4.1 *The stationary distribution $\boldsymbol{\rho}_\lambda$ converges, as $\lambda \to \infty$, to the vector $\boldsymbol{\rho}$ such that*

$$\boldsymbol{\rho}(qA^{-1} - \Theta U(q))^{-1}\Theta U = \mathbf{0}, \qquad \boldsymbol{\rho}\mathbf{1} = 1. \tag{24}$$

where $U(q)$ is the unique solution of

$$\frac{1}{2}\Theta^2 X^2 + DX + (Q - qI) = 0 \tag{25}$$

with eigenvalues of negative real parts, and where $U = U(0)$. Both U and $U(q)$ are generators. For $q > 0$, all eigenvalues of $U(q)$ have strictly negative real parts; for $q = 0$, one eigenvalue of U is equal to 0, the others have strictly negative real parts.

Furthermore,

$$\boldsymbol{\rho}(qA^{-1} - \Theta U(q))^{-1} = c_1\boldsymbol{\nu} \tag{26}$$

for some scalar c_1, where

$$\boldsymbol{\nu}\Theta U = \mathbf{0}, \qquad \boldsymbol{\nu}\mathbf{1} = 1. \tag{27}$$

Proof The matrix Υ_λ defined in (12) does not depend on the behaviour of the fluid queue at level 0, so that Lemma 3.3 applies and the stationary probability vector $\boldsymbol{\rho}_\lambda$ of $\bar{\varphi}_\lambda(t)$ is given by (17).

The matrix $P_{\lambda,0}$ defined in (14) is given here by

$$\begin{aligned} P_{\lambda,0} &= \int_0^\infty q e^{-qu} e^{A(Q-\lambda I)u/\sqrt{\lambda}}\,\mathrm{d}u \\ &= q(\sqrt{\lambda}A + qI - (1/\sqrt{\lambda})AQ)^{-1} \\ &= \frac{1}{\sqrt{\lambda}} qA^{-1}(I + \frac{1}{\sqrt{\lambda}} qA^{-1} - \frac{1}{\lambda} AQA^{-1})^{-1} \\ &= \frac{1}{\sqrt{\lambda}} qA^{-1} + O(1/\lambda) \end{aligned} \tag{28}$$

and similarly

$$\begin{aligned} P_{\lambda,1} &= \int_0^\infty e^{-qu} e^{A(Q-\lambda I)u/\sqrt{\lambda}} \sqrt{\lambda} A\,\mathrm{d}u \\ &= \sqrt{\lambda}(\sqrt{\lambda}A + qI - (1/\sqrt{\lambda})AQ)^{-1}A \\ &= I - \frac{1}{\sqrt{\lambda}} qA^{-1} + O(1/\lambda) \end{aligned} \tag{29}$$

We repeat the proof of [14, Lemma 3.4], replacing Q by $Q - qI$, and obtain that

$$\Psi_\lambda(q) = I + \frac{1}{\sqrt{\lambda}}\Theta U(q) + O(1/\lambda) \tag{30}$$

for $q \geq 0$, where $U(q)$ is as stated in the lemma. Altogether, the transition matrix is

$$\begin{aligned} \Phi_\lambda = {} & [I - (I - \frac{1}{\sqrt{\lambda}} qA^{-1})(I + \frac{1}{\sqrt{\lambda}}\Theta U(q))]^{-1} \\ & \times [\frac{1}{\sqrt{\lambda}} qA^{-1} + (I - \frac{1}{\sqrt{\lambda}} qA^{-1})\frac{1}{\sqrt{\lambda}}\Theta(U - U(q))] + O(1/\lambda) \\ @ = {} & I + (qA^{-1} - \Theta U(q))^{-1}\Theta U + O(1/\sqrt{\lambda}), \end{aligned} \tag{31}$$

which converges to the stochastic matrix

$$\Phi = I + (qA^{-1} - \Theta U(q))^{-1}\Theta U$$

as $\lambda \to \infty$. The matrices Φ and Φ_λ are irreducible and so the stationary probability vector $\boldsymbol{\rho}_\lambda$ of Φ_λ converges to the stationary probability vector $\boldsymbol{\rho}$ of Φ, from which (24) follows.

The remainder of the proof is immediate — observe that (27) is meaningful as U is an irreducible generator. $\square$

Remark 4.2 The matrix $U(q)$ defined in Lemma 4.1 has the following physical interpretation: define $t_x = \inf\{t : X(t) \geq x\}$. we have

$$(e^{U(q)x})_{ij} = \mathrm{P}[t_x < E_q, \varphi(t_x) = j | \varphi(0) = i]$$

where E_q is an exponentially distributed random variable with parameter q. In other words, $U(q)$ is the generator of the Markov process $\{\varphi(t_x)\}$ if the MMBM is killed at the exponential time E_q. Equation (25) is a particular case of [11, Eqn (2.2)].

Lemma 4.3 *The expected time spent by the process* $\{Y_\lambda(t), \bar{\varphi}_\lambda(t)\}$ *in the closed interval* $[0, x]$ *between regeneration points converges, as* $\lambda \to \infty$, *to*

$$M(x) = (qA^{-1} - \Theta U(q))^{-1} (A^{-1} + 2(-K)^{-1}(I - e^{Kx})\Theta^{-1}) \tag{32}$$

for $x \geq 0$, *where*

$$K = \Theta U \Theta^{-1} + 2\Theta^{-2} D. \tag{33}$$

The expected inter-regeneration interval is

$$\boldsymbol{m} = (qA^{-1} - \Theta U(q))^{-1} (A^{-1} + 2(-K)^{-1}\Theta^{-1})\mathbf{1}. \tag{34}$$

Proof We denote by $M_\lambda(x)$ the expected time spent by the fluid queue in $[0, x]$ and we examine $M_\lambda(0)$ first. After a regeneration, the process remains at level 0 either until the timer expires, or until $\bar{\kappa}_\lambda$ switches from 2 to 1. If the timer expires first, the sojourn time at 0 is over, otherwise, the fluid begins to grow and we wait for the level to return to 0. If the return to 0 happens after the timer has expired, the sojourn time at 0 is over, otherwise an additional interval at 0 begins. Thus,

$$\begin{aligned} M_\lambda(0) &= (\sqrt{\lambda}A + qI - (1/\sqrt{\lambda})AQ)^{-1} + P_{\lambda,1}\Psi_\lambda(q)M_\lambda(0) \\ &= (I - P_{\lambda,1}\Psi_\lambda(q))^{-1}(\sqrt{\lambda}A + qI - (1/\sqrt{\lambda})AQ)^{-1} \\ &= (\frac{1}{\sqrt{\lambda}}(qA^{-1} - \Theta U(q)) + O(1/\lambda))^{-1}(\frac{1}{\sqrt{\lambda}}A^{-1} + O(1/\lambda)) \end{aligned}$$

by (29, 30). This converges to $(qA^{-1} - \Theta U(q))^{-1}A^{-1}$ as $\lambda \to \infty$.

For $x > 0$, we have by a similar decomposition

$$M_\lambda(x) = (\sqrt{\lambda}A + qI - (1/\sqrt{\lambda})AQ)^{-1} + P_{\lambda,1}M_f(x) + P_{\lambda,1}\Psi_\lambda(q)M_\lambda(x), \tag{35}$$

where $M_f(x)$ is the matrix of expected time spent in the semi-open interval $(0, x]$ until the next return to 0, irrespective of the timer being off or not at the time of return. We rewrite (35) as

$$\begin{aligned} M_\lambda(x) &= (I - P_{\lambda,1}\Psi_\lambda(q))^{-1} ((\sqrt{\lambda}A + qI - (1/\sqrt{\lambda})AQ)^{-1} + P_{\lambda,1}M_f(x)) \\ &= M_\lambda(0) + (I - P_{\lambda,1}\Psi_\lambda(q))^{-1}P_{\lambda,1}M_f(x). \end{aligned} \tag{36}$$

By (29, 30), we have

$$(I - P_{\lambda,1}\Psi(q))^{-1}P_{\lambda,1} = \sqrt{\lambda}((-U(q))^{-1}\Theta^{-1} + O(1/\sqrt{\lambda})$$

and by [14, Lemma 3.6] and [12, Theorem 3.7],

$$M_f(x) = \frac{2}{\sqrt{\lambda}}(-K)^{-1}(I - e^{Kx})\Theta^{-1} + O(1/\lambda)$$

where K is given by (33). Altogether, this shows that the limit of $M_\lambda(x)$ is given by (32). The proof of (34) is immediate. □

We collect Lemmas 4.1 and 4.3 in the theorem below and obtain two formally different expressions for the stationary distribution of the MMBM with sticky boundary. The first one directly follows from our regenerative process approach, the second is independent of the parameter q. In particular, one may verify that (38) is identical to (7) when there is only one phase.

Theorem 4.4 *The stationary probability distribution function of the MMBM with sticky boundary at zero is given by*

$$\boldsymbol{G}(x) = \gamma_\rho\, \boldsymbol{\rho}(qA^{-1} - \Theta U(q))^{-1}(A^{-1} + 2(-K)^{-1}(I - e^{Kx})\Theta^{-1}), \qquad (37)$$

where $\boldsymbol{\rho}$ is the solution of the system (24) and $\gamma_\rho = 2(\boldsymbol{\rho m})^{-1}$ is the normalization constant.

The distribution is also given by

$$\boldsymbol{G}(x) = \gamma_\nu \boldsymbol{\nu}(A^{-1} + 2(-K)^{-1}(I - e^{Kx})\Theta^{-1}), \qquad (38)$$

independently of q, where $\boldsymbol{\nu}$ is the solution of the system (27) and $\gamma_\nu = (\boldsymbol{\nu}(A^{-1} + 2(-\Theta K)^{-1})\mathbf{1})^{-1}$ is the normalizing constant. □

Proof The process $\{Y_\lambda(t), \bar{\kappa}_\lambda(t), \bar{\varphi}_\lambda(t)\}$ has the same distribution as the process $\{Z_\lambda(\Gamma_\lambda(t)), \kappa_\lambda(\Gamma_\lambda(t)), \varphi_\lambda(\Gamma_\lambda(t))\}$. By [14, Theorem 2.7], $\{Z_\lambda(t), \varphi_\lambda(t)\}$ weakly converges to $\{Z(t), \varphi(t)\}$ and so, by the continuity and convergence properties of $\Gamma(t)$, we find that the finite-dimensional distributions of $\{Y_\lambda(t), \bar{\varphi}_\lambda(t)\}$ converge to the finite-dimensional distribution of $\{Y(t), \bar{\varphi}(t)\}$.

Furthermore, the family $\{Y_\lambda(t), \bar{\varphi}_\lambda(t)\}$ is tight. To see this, we adapt the proof of [14, Theorem 2.6] and use Lemma 3.3.

Finally, we adapt the proof of [13, Theorem 3.1] to conclude that the stationary distribution of $\{Y_\lambda(t), \bar{\varphi}_\lambda(t)\}$ converge to the stationary distribution of $\{Y(t), \bar{\varphi}(t)\}$. Together with Lemmas 4.1 and 4.3, this completes the proof. □

The presence of the factor A^{-1} in the expression for $M(0)$ is easy to understand: the greater a_i, the faster the process leaves level 0 and the smaller the mass at zero for phase i. The marginal distribution of the phase is no longer equal to $\boldsymbol{\alpha}$, as we show in corollary 4.5 below; its proof is immediate and is omitted.

Corollary 4.5 *The marginal distribution of the phase is*

$$\boldsymbol{G}(\infty) = \gamma_\nu \boldsymbol{\nu}(A^{-1} + 2(-\Theta K)^{-1}).$$

□

5 Markov-regenerative MMBM

We revisit here the standard MMBM $\{Z(t), \varphi(t)\}$ defined in (1) and follow our regenerative process approach to determine its stationary distribution. Although expressions are known already for the stationary distribution (Rogers [18], Asmussen [1], Latouche and Nguyen [14]), this new analysis is of independent interest because it is one of the first to analyze the MMBM as a regenerative process. Harrison [9, Chapter 5, Section 4] does treat the regulated Brownian motion with two boundaries as a regenerative process, but we take a different path.

We follow the same steps as in Section 4 and, to avoid confusion with the results there, we use the mark "$*$" in the present section. Thus, $\boldsymbol{\rho}^*_\lambda$ and $M^*_\lambda(x)$ represent, respectively, the stationary distribution of the phase at regeneration epochs, and the expected time in $[0, x]$ between regenerations, for the flip-flop process $\{Z_\lambda(t), \kappa_\lambda(t), \varphi_\lambda(t)\}$ with generator (10).

Lemma 5.1 *As $\lambda \to \infty$, $\boldsymbol{\rho}^*_\lambda$ converges to $\boldsymbol{\rho}^*$ such that*

$$\boldsymbol{\rho}^*(-U(q))^{-1}U = \mathbf{0}, \qquad \boldsymbol{\rho}^*\mathbf{1} = 1, \tag{39}$$

where $U(q)$ is defined in Lemma 4.1. In addition,

$$\boldsymbol{\rho}^*(-U(q))^{-1} = c_2\boldsymbol{\nu}\Theta, \tag{40}$$

for some scalar c_2, where $\boldsymbol{\nu}$ is characterized by (27).

Proof We start from Lemma 3.3 and we repeat the argument in the proof of Lemma 4.1, the only difference being that the matrices $P_{\lambda,0}$ and $P_{\lambda,1}$ are given here by

$$P_{\lambda,0} = q(\lambda I + qI - Q)^{-1} = O(1/\lambda), \tag{41}$$

$$P_{\lambda,1} = \lambda(\lambda I + qI - Q)^{-1} = I + O(1/\lambda), \tag{42}$$

so that the matrix Φ_λ from (16) is

$$\Phi_\lambda = I - U(q)^{-1}U + O(1/\sqrt{\lambda})$$

and converge to $\Phi = I - U(q)^{-1}U$. The remainder of the proof is straightforward. $\square$

Our next step is to determine the expected time spent in $[0, x]$ during a regeneration interval, and then collect the pieces in Theorem 5.3.

Lemma 5.2 *The expected time spent by the MMBM in level 0 between regeneration points is 0. The time spent in $[0, x]$ (or equivalently in $(0, x]$) is*

$$M^*(x) = 2(-U(q))^{-1}\Theta^{-1}(-K)^{-1}(I - e^{Kx})\Theta^{-1} \qquad \text{for } x \geq 0. \tag{43}$$

The expected length of an interval between regenerations is

$$\boldsymbol{m}^* = 2(-U(q))^{-1}\Theta^{-1}(-K)^{-1}\Theta^{-1}\mathbf{1}. \tag{44}$$

Proof We follow the same steps as in the proof of Lemma 4.3. The expected time at level zero is

$$\begin{aligned} M^*_\lambda(0) &= (\lambda I + qI - Q)^{-1} + P_{\lambda,1}\Psi_\lambda(q)M^*_\lambda(0) \\ &= (I - P_{\lambda,1}\Psi_\lambda(q))^{-1}(\lambda I + qI - Q)^{-1} \\ &= \frac{1}{\sqrt{\lambda}}(-U(q))^{-1}\Theta^{-1} + O(1/\lambda) \end{aligned} \tag{45}$$

and so, $\lim_{\lambda\to\infty} M^*_\lambda(0) = 0$. For strictly positive x, we have

$$M^*_\lambda(x) = M^*_\lambda(0) + (I - P_{\lambda,1}\Psi(q))^{-1}P_{\lambda,1}M^*_f(x)$$

instead of (36), and (43) follows after simple manipulations. The proof of (44) is immediate. $\square$

The next theorem directly follows from Lemmas 5.1 and 5.2 and is given without proof.

Theorem 5.3 *The stationary probability distribution function of the regulated MMBM is given by*

$$\boldsymbol{G}^*(x) = \gamma^*_\rho\,\boldsymbol{\rho}^*(-U(q))^{-1}\Theta^{-1}(-K)^{-1}(I - e^{Kx})\Theta^{-1}, \tag{46}$$

where $\boldsymbol{\rho}^*$ is the solution of $\boldsymbol{\rho}^*(-U(q))^{-1}U = \mathbf{0}$, $\boldsymbol{\rho}^*\mathbf{1} = 1$, and $\gamma^*_\rho = 2(\boldsymbol{\rho}^*\boldsymbol{m}^*)^{-1}$ is the normalizing constant.

It is also given by

$$\boldsymbol{G}^*(x) = \gamma^*_\nu\boldsymbol{\nu}(-K)^{-1}(I - e^{Kx})\Theta^{-1}, \tag{47}$$

independently of q, where $\boldsymbol{\nu}$ is the solution of the system $\boldsymbol{\nu}\Theta U = \mathbf{0}$, $\boldsymbol{\nu}\mathbf{1} = 1$, and $\gamma^*_\nu = (\boldsymbol{\nu}(-K)^{-1}\Theta^{-1}\mathbf{1})^{-1}$. $\square$

The vector $\boldsymbol{\nu}$ is proportional to the vector $\boldsymbol{\zeta}_1$ defined in [14, Theorem 3.7] and so the expression (47) is nearly identical to the one given there.

6 Observations

The equations (38, 47) have the advantage over (37, 46) of being independent of the artificial parameter q. On the other hand, the vectors $\boldsymbol{\rho}$ and $\boldsymbol{\rho}^*$ have the physical meaning of being the stationary distribution of the phase at epochs of regeneration, while the interpretation of $\boldsymbol{\nu}$ is not as clear, as we discuss below.

We define the set $\mathcal{E} = \{\theta_n : n \geq 0\}$ of regeneration epochs, and we partition it into three disjoint subsets:

$$\begin{aligned}
\mathcal{E}_0 &= \{\theta_n : \theta_n = \theta_{n-1} + \Delta_n < \tau_{n-1}, n \geq 1\} \\
\mathcal{E}_1 &= \{\theta_n : \theta_n = \theta_{n-1} + \Delta_n > \tau_{n-1}, n \geq 1\} \\
\mathcal{E}_a &= \{\theta_n : \theta_n > \theta_{n-1} + \Delta_n, n \geq 1\}.
\end{aligned}$$

If θ_n is in $\mathcal{E}_0$ or in $\mathcal{E}_1$, it means that the fluid is equal to zero when the timer expires; in the first case, it has not left level 0 at all between θ_{n-1} and θ_n, in the second case, the fluid has left level 0 and has returned there, possibly several times. If θ_n is in $\mathcal{E}_a$, then the fluid is above level 0 when the timer expires. To keep the notation simple, we do not indicate that these sets depend on λ.

We also define $\mathcal{E}^+ = \{\theta_n^+\}$ to be the set of *all* epochs when the fluid hits level 0 from above: starting from $\theta_0^+ = 0$, we define

$$\begin{aligned}
\tau_n^+ &= \inf\{t > \theta_n^+ : \varphi_\lambda(t) \in \mathcal{S}_+\}, \\
\theta_{n+1}^+ &= \inf\{t > \tau_n^+ : Z_\lambda(t) = 0\}.
\end{aligned}$$

Clearly, $\mathcal{E}_a \subset \mathcal{E}^+$, and $\mathcal{E}_b$ defined as $\mathcal{E}_b = \mathcal{E}^+ \setminus \mathcal{E}_a$ is the set of all epochs when the process returns to 0 from above before the expiration of the timer.

By definition, $\boldsymbol{\rho}_\lambda$ (as well as $\boldsymbol{\rho}^*_\lambda$) is the limiting distribution of $\varphi_\lambda(t)$ as t goes to infinity by taking values in $\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_a$, while we see from (18) that $\boldsymbol{\nu}_\lambda$ is the limiting distribution as t goes to infinity by taking values in $\mathcal{E}_0 \cup \mathcal{E}^+ = \mathcal{E}_0 \cup \mathcal{E}_b \cup \mathcal{E}_a$. This observation provides us with a physical interpretation for (17): we rewrite that equation as

$$\boldsymbol{\rho}_\lambda (I - P_{\lambda,1} \Psi_\lambda(q))^{-1} = c \boldsymbol{\nu}_\lambda$$

and we note that $(I - P_{\lambda,1} \Psi_\lambda(q))^{-1}$ is the matrix of expected number of returns to level 0 at epochs in $\mathcal{E}_b$ between two successive regeneration points.

We now focus on the traditional MMBM analyzed in Section 5, and we compare $\boldsymbol{\nu}_\lambda$ and $\boldsymbol{\rho}^*_\lambda$. In this case, $\mathcal{E}_0$ and $\mathcal{E}_1$ vanish as λ grows bigger, the vector $\boldsymbol{\nu}_\lambda$ becomes more like the stationary distribution of the phase at *all epochs* when the fluid returns to level 0, irrespective of the timer, while $\boldsymbol{\rho}^*_\lambda$ becomes more like the stationary distribution at the *subset* of those epochs when we have an actual regeneration. In the limit, the interpretation of $\boldsymbol{\nu}_\lambda$ may not be given as such to $\boldsymbol{\nu}$ due to the instantaneous repeated hits at the boundary by the Brownian motion, once it reaches level 0.

The vectors $\boldsymbol{\rho}^*$ and $\boldsymbol{\nu}$ are both related to the regulator $R(t)$ of $\{X(t)\}$. Recall that $R(t) = |\inf_{0\le s\le t} X(s)|$ is split into the sub-regulators defined in (3). It is shown in Asmussen and Kella [2] that $r_i(t)/t$ converges almost surely as $t \to \infty$, and we define

$$\ell_i = \lim_{t\to\infty} r_i(t)/t. \tag{48}$$

Similarly, the regulator $R_\lambda(t) = |\inf_{0\le s\le t} X_\lambda(s)|$ of the fluid queue is split into the sub-regulators $r_{\lambda,i}(t)$ in (21) and we define $\ell_{\lambda,i} = \lim_{t\to\infty} r_{\lambda,i}(t)/t$.

We proved in [13, 14] the weak convergence of $\{Z_\lambda(t), \varphi_\lambda(t)\}$ to $\{Z(t), \varphi(t)\}$. In consequence, the functions $R_\lambda(t)$, $t \ge 0$, weakly converge to $R(t)$ and the vectors $\boldsymbol{\ell}_\lambda$ converge to $\boldsymbol{\ell}$, as $\lambda \to \infty$.

The function $r_{\lambda,i}(t)$ increases at the rate $|\mu_i - \sqrt{\lambda}\sigma_i|$ during those intervals of time when $(Z_\lambda(t), \varphi_\lambda(t)) = (0, i)$, and so

$$\begin{aligned}
\ell_{\lambda,i} &= (\sigma_i\sqrt{\lambda} - \mu_i)(\boldsymbol{\rho}^*_\lambda \boldsymbol{m}^*_\lambda)^{-1}(\boldsymbol{\rho}^*_\lambda M^*_\lambda(0))_i \\
&= (\sigma_i\sqrt{\lambda} - \mu_i)(\boldsymbol{\rho}^*\boldsymbol{m}^* + O(1/\sqrt{\lambda}))^{-1}(\frac{1}{\sqrt{\lambda}}\boldsymbol{\rho}^*(-U(q))^{-1}\Theta^{-1} + O(1/\lambda))_i \\
&= (\boldsymbol{\rho}^*\boldsymbol{m}^*)^{-1}(\boldsymbol{\rho}^*(-U(q))^{-1})_i + O(1/\sqrt{\lambda})
\end{aligned}$$

from which we obtain

$$\begin{aligned}
\boldsymbol{\ell} &= (\boldsymbol{\rho}^*\boldsymbol{m}^*)^{-1}(\boldsymbol{\rho}^*(-U(q))^{-1}) && (49)\\
&= c_3\boldsymbol{\nu}\Theta && (50)
\end{aligned}$$

for some scalar c_3 by (40). This provides us with another expression for the stationary distribution of MMBMs.

Corollary 6.1 *The stationary probability distribution function of the MMBM is given by*

$$\boldsymbol{G}^*(x) = 2\boldsymbol{\ell}\Theta^{-1}(-K)^{-1}(I - e^{Kx})\Theta^{-1} \tag{51}$$

where $\boldsymbol{\ell}$ is defined in (48). The vector $\boldsymbol{\ell}$ is the solution of the linear system

$$\boldsymbol{\ell}U = \boldsymbol{0}, \qquad 2\boldsymbol{\ell}\Theta^{-1}(-K)^{-1}\Theta^{-1}\boldsymbol{1} = 1. \tag{52}$$

Proof Equation (51) is a direct consequence of (46) and of the relation (49) between $\boldsymbol{\ell}$ and $\boldsymbol{\rho}^*$. Also, by (50), we find that $\boldsymbol{\ell} U = \mathbf{0}$ since $\boldsymbol{\nu}\Theta U = \mathbf{0}$ by (27). Finally, we use the normalizing equation $\boldsymbol{G}^*(\infty)\mathbf{1} = 1$ and the proof is complete. $\square$

One last representation of the stationary distribution $\boldsymbol{G}^*$ establishes a direct connection with the stationary distribution $\boldsymbol{\alpha}$ of the Markov process $\{\varphi(t)\}$ with generator Q.

Corollary 6.2 *The stationary probability distribution function of the MMBM is given by*

$$\boldsymbol{G}^*(x) = \boldsymbol{\alpha}\Theta(I - e^{Kx})\Theta^{-1} \tag{53}$$

where $\boldsymbol{\alpha}$ is the solution of the system $\boldsymbol{\alpha} Q = \mathbf{0}$, $\boldsymbol{\alpha}\mathbf{1} = 1$.

Proof Obviously, the marginal distribution of the phase is $\boldsymbol{\alpha}$, so that by (47)

$$\boldsymbol{\alpha} = \boldsymbol{G}^*(\infty) = \gamma_\nu^* \boldsymbol{\nu}(-K)^{-1}\Theta^{-1} \tag{54}$$

and (47) may be rewritten as (53). $\square$

Corollary 6.2 may also be proved by a purely algebraic argument. We give it below as it illustrates the intricate interconnection between different matrices. We proceed through the sequence of equations

$$\begin{array}{rrl} & \boldsymbol{\alpha} Q = \mathbf{0} & \\ \Leftrightarrow & \boldsymbol{\alpha}(\Theta^2 U^2 + 2DU) = \mathbf{0} & \text{by (25) evaluated at } q = 0, \\ \Leftrightarrow & \boldsymbol{\alpha}\Theta K \Theta U = \mathbf{0} & \text{by (33)} \\ \Leftrightarrow & \boldsymbol{\alpha}\Theta K = c_4 \boldsymbol{\nu} & \end{array}$$

for some scalar c_4 by (27). Therefore, (47) becomes

$$\boldsymbol{G}^*(x) = c_5 \boldsymbol{\alpha}\Theta(I - e^{Kx})\Theta^{-1}$$

for some scalar c_5, and it is easily seen that $c_5 = 1$ since $\boldsymbol{G}^*(\infty)\mathbf{1} = 1$.

Remark 6.3 Clearly, the stationary distribution of MMBMs may be expressed under many different guises, even without counting the ones based on the time-reversed process, as in [1, 18]. We find the matrix $(I - e^{Kx})\Theta^{-1}$ in each case, pre-multiplied by vectors which depend on the behavior of the process at the boundary.

The connections (51) with the vector $\boldsymbol{\ell}$, and (53) with the vector $\boldsymbol{\alpha}$ crucially depend on the evolution of the phase being independent of the fluid level. Indeed, both $\boldsymbol{\ell}$ and $\boldsymbol{\alpha}$ are defined by the unrestricted MMBM:

(a) The vector $\boldsymbol{\alpha}$ is the stationary marginal distribution of the phase when its evolution is governed by the matrix Q and is not modified in any way; this is the key to the proof of Corollary 6.2.

(b) The vector $\boldsymbol{\ell}$ is defined in (48) as the rate of increase of the regulator in the absence of any barrier. Corollary 6.1 requires that the stationary distribution at regenerations be related to $\boldsymbol{\ell}$ through (49), which is not true of the process analyzed in Section 4 and the one defined in the next section.

7 Resampling the phase

In Section 4, we slow down the evolution of the process at level 0 and we use different factors a_i for different phases, but the behavior of the phase is not otherwise modified. We go one step further now and allow for more general perturbations. In the generator (23) of the fluid queue $\{Y_\lambda(t), \bar{\kappa}_\lambda(t), \bar{\varphi}_\lambda(t)\}$ at level 0, transitions of $\bar{\kappa}_\lambda$ from 2 to 1 occur at rates proportional to $\sqrt{\lambda}$ while transitions of $\bar{\varphi}_\lambda$ occur at much smaller rates of order $1/\sqrt{\lambda}$. We shall now assume that both $\bar{\kappa}_\lambda$ and $\bar{\varphi}_\lambda$ may evolve at rates proportional to $\sqrt{\lambda}$.

We define a new family $\{\widetilde{Y}_\lambda(t), \widetilde{\kappa}_\lambda(t), \widetilde{\varphi}_\lambda(t)\}$ of fluid queues with generator Q^*_λ given in (10) when $\widetilde{Y}_\lambda > 0$, and generator

$$\widetilde{Q}_{\lambda,0} = \begin{bmatrix} \sqrt{\lambda} A & (1/\sqrt{\lambda})\widetilde{Q} + \sqrt{\lambda}\widetilde{A} \end{bmatrix} \tag{55}$$

when $\widetilde{Y}_\lambda = 0$. That is, simultaneous transitions are possible from $(\bar{\kappa}_\lambda, \bar{\varphi}_\lambda) = (2, i)$ to $(1, j)$ for $i \neq j$ at the rate $\sqrt{\lambda} A_{ij}$. In (23), A is a diagonal matrix, $\widetilde{A} = -A$, and $\widetilde{Q} = AQ$.

Assumption 7.1 *We assume that $A \geq 0$, $\widetilde{A}_{ij} \geq 0$ for $i \neq j$, $\widetilde{A}_{ii} < 0$, i, j in $\mathcal{M}$, and that $A + \widetilde{A}$ is an irreducible generator. The matrix $\widetilde{Q}$ is such that $(1/\sqrt{\lambda})\widetilde{Q} + \sqrt{\lambda}\widetilde{A}$ is an irreducible generator for λ large enough.*

In consequence, the matrix B defined as $B = (-\widetilde{A})^{-1} A$ is stochastic and irreducible, and we denote its stationary probability vector as $\boldsymbol{\beta}$. The assumption that $A + \widetilde{A}$ is irreducible is a significant restriction: by contrast, $A + \widetilde{A} = 0$ for the transition matrix (23), and B is the identity matrix. We make this assumption so as to simplify the presentation of the process and to let its major feature stand out.

The matrix $\widetilde{Q}$ plays a minor role since the speed of changes induced by $(1/\sqrt{\lambda})\widetilde{Q}$ is negligible with respect to $\sqrt{\lambda}$. Actually, our expression in Theorem 7.5 for the limiting stationary distribution does not depend on $\widetilde{Q}$.

Because of the additional mixing of the phases allowed by the matrices A and $\widetilde{A}$, we have a more complex transformation than the simple change of clock in Section 4. Away from 0, $\{\widetilde{Y}_\lambda(t), \widetilde{\varphi}_\lambda(t)\}$ behaves exactly like $\{Z_\lambda(t), \varphi_\lambda(t)\}$ but at level 0, as λ grows bigger, the evolution of the phase is controlled mostly through the matrices $\sqrt{\lambda}\widetilde{A}$ and $\sqrt{\lambda}A$ and, in first approximation, the distribution of the phase is repeatedly transformed by the transition matrix B upon each visit to the boundary.

Lemma 7.2 *The stationary distribution of $\widetilde{\varphi}_\lambda$ at epochs of regeneration converges as $\lambda \to \infty$ to $\widetilde{\boldsymbol{\rho}}$ with*

$$\widetilde{\boldsymbol{\rho}} = \widetilde{\gamma}\boldsymbol{\beta}(q(-\widetilde{A})^{-1} + \Theta(U - U(q))), \tag{56}$$

where $\boldsymbol{\beta}$ is the stationary probability vector of B and

$$\widetilde{\gamma} = (\boldsymbol{\beta}(q(-\widetilde{A})^{-1} - \Theta U(q))\mathbf{1})^{-1} \tag{57}$$

is the normalization constant.

Proof The transition matrix at epochs of regeneration is given by (19), where

$$\begin{aligned} P_{\lambda,0} &= q(-\sqrt{\lambda}\widetilde{A} + qI - (1/\sqrt{\lambda})\widetilde{Q})^{-1} \\ &= \frac{1}{\sqrt{\lambda}} q(-\widetilde{A})^{-1} + O(1/\lambda) \end{aligned}$$

$$\begin{aligned} P_{\lambda,1} &= \sqrt{\lambda}(-\sqrt{\lambda}\widetilde{A} + qI - (1/\sqrt{\lambda})\widetilde{Q})^{-1}A \\ &= B + \frac{1}{\sqrt{\lambda}} q\widetilde{A}^{-1}B + O(1/\lambda). \end{aligned}$$

Taking the limit as $\lambda \to \infty$ on both sides of (19), we find that the limit Φ of Φ_λ satisfies the equation $\Phi = B\Phi$ where B is stochastic. This shows that Φ is of rank one, and that

$$\Phi = \mathbf{1} \cdot \widetilde{\boldsymbol{\rho}} \tag{58}$$

for some vector $\widetilde{\boldsymbol{\rho}}$. The matrix Φ is stochastic, and so $\widetilde{\boldsymbol{\rho}}$ is its stationary probability vector. Thus,

$$\Phi_\lambda = \mathbf{1} \cdot \widetilde{\boldsymbol{\rho}} + \frac{1}{\sqrt{\lambda}}\Phi' + O(1/\lambda)$$

for some matrix Φ' and, by equating the coefficients of $1/\sqrt{\lambda}$ on both sides of (19), we get

$$\Phi' = q(-\widetilde{A})^{-1} + B\Theta(U - U(q)) + B\Phi' + B\Theta U(q)\mathbf{1} \cdot \widetilde{\boldsymbol{\rho}} + q\widetilde{A}^{-1}B\mathbf{1} \cdot \widetilde{\boldsymbol{\rho}}.$$

since $B\mathbf{1} = \mathbf{1}$. We pre-multiply both sides by $\boldsymbol{\beta}$ and obtain

$$\mathbf{0} = \boldsymbol{\beta}(q(-\widetilde{A})^{-1} + \Theta(U - U(q))) + (\boldsymbol{\beta}\Theta U(q)\mathbf{1} + q\boldsymbol{\beta}\widetilde{A}^{-1}\mathbf{1})\widetilde{\boldsymbol{\rho}}$$

from which (56, 57) follow. □

Lemma 7.3 *In the limit as $\lambda \to \infty$, the expected time spent in $[0, x]$ between regeneration points is*

$$\widetilde{M}(x) = \widetilde{\gamma}\,\mathbf{1} \cdot \boldsymbol{\beta}(-\widetilde{A}^{-1} + 2(-K)^{-1}(I - e^{Kx})\Theta^{-1}) \tag{59}$$

for $x \geq 0$, and

$$\widetilde{\boldsymbol{m}} = \widetilde{\gamma}\,(\boldsymbol{\beta}(-\widetilde{A}^{-1} + 2(-K)^{-1}\Theta^{-1})\mathbf{1})\,\mathbf{1}. \tag{60}$$

Proof We decompose $\widetilde{M}_\lambda(x)$ as in Lemmas 4.3 and 5.2:

$$\begin{aligned}
\widetilde{M}_\lambda(x) &= (-\sqrt{\lambda}\widetilde{A} + qI - \frac{1}{\sqrt{\lambda}}\widetilde{Q})^{-1} + P_{\lambda,1}M_f(x) + P_{\lambda,1}\Psi_\lambda(q)\widetilde{M}_\lambda(x) \\
&= \frac{1}{\sqrt{\lambda}}(-\widetilde{A})^{-1} + (B + \frac{1}{\sqrt{\lambda}}q\widetilde{A}^{-1}M_f(x)) \\
&\quad + (B + \frac{1}{\sqrt{\lambda}}q\widetilde{A}^{-1}(I + \frac{1}{\sqrt{\lambda}}\Theta U(q))\widetilde{M}_\lambda(x) + O(1/\lambda).
\end{aligned}$$

In the limit, $\widetilde{M}_\lambda(x)$ converges to a solution of $\widetilde{M}(x) = B\widetilde{M}(x)$, so that

$$\lim_{\lambda\to\infty} \widetilde{M}_\lambda(x) = \mathbf{1} \cdot \widetilde{\boldsymbol{\mu}}(x)$$

for some vector $\widetilde{\boldsymbol{\mu}}(x)$ which needs to be determined. We proceed just like we did in the proof of Lemma 7.2, and obtain

$$\widetilde{\boldsymbol{\mu}}(x) = \widetilde{\gamma}\boldsymbol{\beta}\,(A^{-1} + 2(-K)^{-1}(I - e^{Kx})\Theta^{-1}).$$

The proof of (60) is immediate. □

Remark 7.4 We observe in (58) the effect of the Brownian motion jiggle at the boundary: by the time the exponential timer is off, the process will have hit level 0 so often that the phase at the next regeneration epoch will be independent of the phase at the last one.

The same effect is at work in (59): after a regeneration point, the phase will be repeatedly re-sampled through the matrix B, so often that the expected length of any interval between regeneration points is independent of the phase at the end of the previous interval, and depends only on the stationary distribution of B.

In short, we refer to the limit as a process with sticky boundary and resampling of the phase at level zero.

Theorem 7.5 *The stationary probability distribution function of the process* $\{\widetilde{Y}(t), \widetilde{\varphi}(t)\}$ *with sticky boundary and resampling of the phase at zero is given by*

$$\widetilde{\boldsymbol{G}}(x) = \widetilde{\gamma}\boldsymbol{\beta}(-\widetilde{A}^{-1} + 2(-K)^{-1}(I - e^{Kx})\Theta^{-1}), \tag{61}$$

independently of q*, where* $\boldsymbol{\beta}$ *is the stationary probability vector of* B *and* $\widetilde{\gamma}$ *is given in (57).*

The marginal distribution of the phase is $\widetilde{\boldsymbol{G}}(\infty) = \widetilde{\gamma}\boldsymbol{\beta}(-\widetilde{A}^{-1}+2(-K)^{-1}\Theta^{-1})$.

Proof By Lemmas 7.2 and 7.3,

$$\begin{aligned}\widetilde{\boldsymbol{G}}(x) &= \widetilde{\gamma}\,\boldsymbol{\beta}(-q\widetilde{A}^{-1} + \Theta(U - U(q)))\,\widetilde{\gamma}\mathbf{1}\cdot\boldsymbol{\beta}(-\widetilde{A}^{-1} + 2(-K)^{-1}(I - e^{Kx})\Theta^{-1}) \\ &= \widetilde{\gamma}\,\boldsymbol{\beta}(-\widetilde{A}^{-1} + 2(-K)^{-1}(I - e^{Kx})\Theta^{-1})\end{aligned}$$

after reorganization of some factors, and using the relation $U\mathbf{1} = \mathbf{0}$. □

8 Concluding remarks

To the best of our knowledge, these are the first results on MMBMs where the evolution of the phase φ may depend on the level, with the exception of Chen *et al.* [6]: the authors consider MMBMs with level-dependent, piecewise constant, fluid rates and obtain the stationary distribution by numerically solving systems of partial differential equations.

Our regenerative approach to the analysis of regulated MMBMs clearly shows great promise in allowing more complex assumptions than has been the case until now. We have demonstrated this on two specific cases of reactive boundaries in Sections 4 and 7 but other examples easily come to mind, as in Latouche and Nguyen [12].

In each case covered here, the stationary distribution is easily calculated once the matrices U and $U(q)$ are determined. Extremely efficient algorithms exist to solve the matrix equation (25), such as those in Latouche and Nguyen [14], and Nguyen and Poloni [15], and so the question of numerically obtaining these distributions is not an issue.

There is a striking difference between the "traditional" process analyzed in Section 5 and the two processes with sticky boundary: the representation of the stationary distribution in Corollaries 6.1 and 6.2 hold in the first case but not in the other two. Instead, the importance of the distribution at epochs of regeneration is made more manifest, observe that the vector $\boldsymbol{\nu}$ in (38) and $\boldsymbol{\beta}$ in (61) are related in the same manner to the distribution at epochs of regeneration: (26) may be written as

$$c_1\boldsymbol{\nu}A^{-1} = \boldsymbol{\rho}M(0)$$

and (56) as

$$c_6\boldsymbol{\beta}(-\widetilde{A})^{-1} = \widetilde{\boldsymbol{\rho}}\widetilde{M}(0)$$

for some scalar c_6. In both equations, the ith component of the vector in the right-hand side is the expected time spent in phase i at level 0 between two regeneration points.

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